$\boxed{\text{Prove that a set } S \text{ is open if and only if each point in } S \text{ is an interior point.}}$

How to prove?

- Know the type of the problem: This type of problem requires the two directions proofs; those are $p \Rightarrow q$ and $q \Rightarrow p$.

- Know how to prove things: For each direction requires the knowledge of how to prove in mathematics; those are direct proof, contraposition, contradiction, contraexample and induction.

- Know the equivalent properties of each statements: Think or write of all properties that make the statements p and q valid. In this case, we use the definition. Indeed,

$$\begin{aligned} \{S \text{ is an open set}\} &\Leftrightarrow \{S \text{ contains non of its boundary points}\} \\ &\Leftrightarrow \left\{\begin{array}{c}\text{each point in } S \text{ is either an} \\ \text{interior point or exterior point}\end{array}\right\} \\ \left\{\begin{array}{c}\text{each point in } S \\ \text{is an interior point}\end{array}\right\} &\Leftrightarrow \left\{\begin{array}{c}\text{For each point in } S \text{ there is some neighborhood} \\ \text{of that point that contains only points of } S\end{array}\right\} \end{aligned}$$

- Translating the statements to mathematical notations if possible and vice versa: This step depends on the students. Some are easy to understand with statements while others with mathematical notations. This step requires the understanding of mathematics and English. Students have to learn how to do this.

$$\begin{aligned} \{S \text{ is an open set}\} &\Leftrightarrow \{\forall z \in S, z \notin bd(S)\} \\ &\Leftrightarrow \{\forall z \in S, z \in int(S) \sqcup ext(S)\} \\ \{\forall z \in S, z \in int(S)\} &\Leftrightarrow \{\forall z \in S, \exists \text{ neighborhood } U_z \in S \text{ such that} U_z \subset S\} \end{aligned}$$

- Find the connections between the two statements: The interior point is the connection between the two statements and the exterior point is what we want to take out.

- Start to prove mathematically: We have to know which statement will be the hypothesis and which statement will be the result we want. It is important and the method is to connect the result we want to the hypothesis we have. In each step/line to another step/line, we have to check the mathematical logic using all the necessary definitions/properties we have.

Now, we give two types of proof to the problem.

Proof. (by description) Suppose that S is an open set then each point $z \in S$ is not a boundary point by definition. This follows that z is either an interior point of S or exterior point of S. But z cannot be an exterior point because there is no neighborhood of z containing no point of S since at least $z \in S$. Thus, each point in S is an interior point. On the other hand, if each point $z \in S$ is an interior point then z is not a boundary point by definition. It follows that S contains no boundary points and thus is an open set. ∎

Proof. (by mean of mathematical notations) We have

$$\begin{aligned} S \text{ is an open set} &\Leftrightarrow \forall z \in S, z \notin bd(S) \\ &\Leftrightarrow \forall z \in S, z \text{ is either in } int(S) \text{ or } ext(S) \end{aligned}$$

Suppose that $z \in ext(S)$, then there is some neighborhood of z containing no point of S. This contradicts to $z \in S$. Thus

$$S \text{ is an open set} \Leftrightarrow \forall z \in S, z \in int(S)$$

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